Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel International Advanced Level

Time 1 hour 30 minutes

Paper reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over ▶

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1. $$\mathbf{M} = \begin{pmatrix} 3x & 7 \\ 4x+1 & 2-x \end{pmatrix}$$

Find the range of values of x for which the determinant of the matrix **M** is positive.

det(M) > 0

(5)

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

$\det(M) = (3x)(2-x) - (7)(4x+1)$

$= (6x - 3x^2) - (28x + 7)$

$= 6x - 3x^2 - 28x - 7$

$= -3x^2 - 22x - 7$

$\det(M) = -3x^2 - 22x - 7$

Since $\det(M) > 0$, $-3x^2 - 22x - 7 > 0$

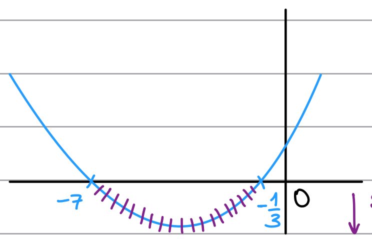
solve quadratic inequality

$-3x^2 - 22x - 7 > 0$

$3x^2 + 22x + 7 < 0$

$(3x+1)(x+7) < 0$

$x = -\frac{1}{3}$ or $x = -7$



since ... < 0
shade part of quadratic below x-axis

$\boxed{-7 < x < -\frac{1}{3}}$

2. The complex numbers z_1 and z_2 are given by

$$z_1 = 3 + 5\text{i} \quad \text{and} \quad z_2 = -2 + 6\text{i}$$

(a) Show z_1 and z_2 on a single Argand diagram.

(2)

(b) **Without using your calculator and showing all stages of your working,**

(i) determine the value of $|z_1|$

(1)

(ii) express $\frac{z_1}{z_2}$ in the form $a + b\text{i}$, where a and b are fully simplified fractions.

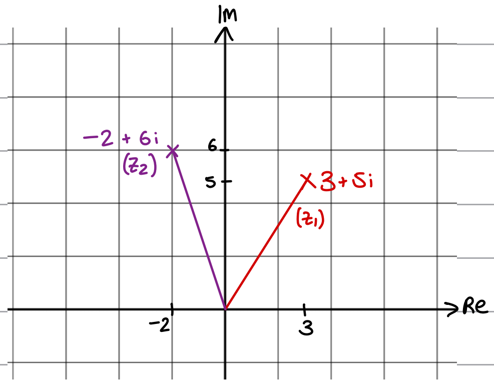
(3)

(c) Hence determine the value of $\arg \frac{z_1}{z_2}$

Give your answer in radians to 2 decimal places.

(2)

a.



bi. $|z_1| = |3+5i|$

$= \sqrt{(3)^2 + (5)^2}$

$= \sqrt{34}$

$\boxed{z_1 = \sqrt{34}}$

multiply top and bottom by complex conjugate of denominator

ii. $\frac{z_1}{z_2} = \frac{3+5i}{-2+6i} \times \frac{-2-6i}{-2-6i} = \frac{(3+5i)(-2-6i)}{(-2+6i)(-2-6i)}$

$= \frac{-6-18i-10i-30i^2}{4+12i-12i-36i^2}$

$= \frac{-6-28i-30(-1)}{4-36(-1)}$

P 7 1 0 9 9 A 0 4 3 6

Question 2 continued

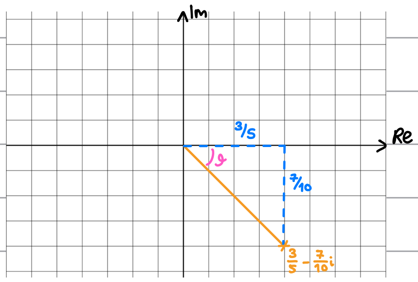
$= \dfrac{-6-28i+30}{4+36}$

$= \dfrac{24-28i}{40}$

$= \dfrac{24}{40} - \dfrac{28}{40}i = \dfrac{3}{5} - \dfrac{7}{10}i$

$\boxed{\dfrac{3}{5} - \dfrac{7}{10}i}$

c. $\arg\left(\dfrac{z_1}{z_2}\right) = \arg\left(\dfrac{3}{5} - \dfrac{7}{10}i\right)$ using part (bii)



$\vartheta = \tan^{-1}\left(\dfrac{7/10}{3/5}\right)$

$\arg(z) = -\vartheta = -\tan^{-1}\left(\dfrac{7/10}{3/5}\right)$

$\arg(z) = -0.862170054 7^c$

$\boxed{\vartheta = -0.86^c \text{ (2d.p.)}}$

3. The parabola C has equation $y^2 = 18x$

The point S is the focus of C

(a) Write down the coordinates of S

(1)

The point P, with $y > 0$, lies on C

The shortest distance from P to the directrix of C is 9 units.

(b) Determine the exact perimeter of the triangle OPS, where O is the origin.

Give your answer in simplest form.

(4)

a. $y^2 = 18x$

$y = 4\left(\frac{18}{4}\right)x \qquad a = \frac{18}{4}$

foci $= (a, 0)$

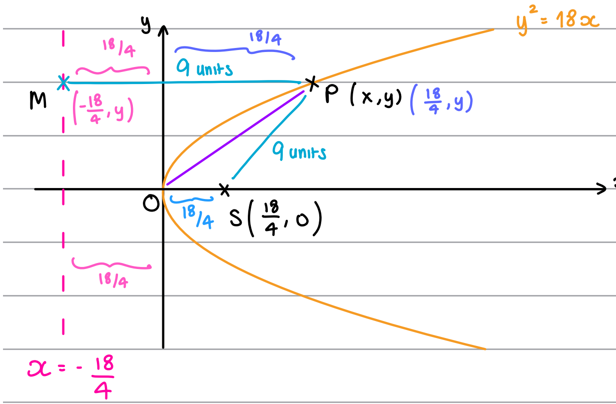
$= \left(\frac{18}{4}, 0\right)$

$S = \left(\frac{18}{4}, 0\right)$

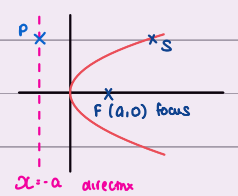
Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

b. Directrix eq^n : $x = \frac{-18}{4}$



Focus-Directrix Property :



$\|FS\| = \|SP\|$

P is point on directrix

S is point on parabola

Due to focus - directrix property, $\|SP\| = \|SM\|$

Q states $\|PM\| = 9 \qquad \therefore \|PM\| = \|SP\| = 9$

distance from y-axis to directrix is $\frac{18}{4}$. (coord M to y-axis)

$\therefore$ distance from y-axis to P = $\frac{18}{4}$ units.

$\therefore$ x-coord. of P is $\frac{18}{4}$

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Question 3 continued

we know $\|SP\| = 9$ and know x-coord of P so can use distance formulae to solve for y-coord of P.

$$\|SP\| = \sqrt{\left(\frac{18}{4} - \frac{18}{4}\right)^2 + (y-0)^2} = 9$$

$$\sqrt{(0)^2 + (y)^2} = 9$$

$$\cancel{\sqrt{y^2}} = 9$$

$$y = 9$$

$\therefore P\left(\frac{18}{4}, 9\right)$

$$\|OP\| = \sqrt{\left(\frac{18}{4} - 0\right)^2 + (9-0)^2}$$

$$= \frac{9\sqrt{5}}{2}$$

Perimetre OPS $= \|OP\| + \|PS\| + \|SO\|$

$$= \frac{9\sqrt{5}}{2} + 9 + \frac{18}{4}$$

$$= \frac{27 + 9\sqrt{5}}{2}$$

$$\boxed{\therefore \text{Perimetre OPS} = \frac{27 + 9\sqrt{5}}{2}}$$

☆ After solved, we realise the diagram is slightly incorrect - S is directly beneath P however there is no way of knowing until diagram is drawn and coord. of P and S are found out.

4. The equation

$$x^4 + Ax^3 + Bx^2 + Cx + 225 = 0$$

where A, B and C are real constants, has

- a complex root 4 + 3i
- a repeated positive real root

(a) Write down the other complex root of this equation. **(1)**

(b) Hence determine a quadratic factor of $x^4 + Ax^3 + Bx^2 + Cx + 225$ **(2)**

(c) Deduce the real root of the equation. **(2)**

(d) Hence determine the value of each of the constants A, B and C **(3)**

a. For a cubic eqⁿ with form: $ax^3 + bx^2 + cx + d$, there are 3 roots.
There are 2 possibilities: 3 real roots
1 real root and 2 complex roots (conjugate pair)

The other complex root must be the complex conjugate of $(4+3i)$
(flip sign of imaginary component

$\boxed{4-3i}$

b. $(x-(4+3i))(x-(4-3i))$

$(x-4-3i)(x-4+3i)$

$= x^2 - 4x + \cancel{3ix} - 4x + 16 - \cancel{12i} - \cancel{3ix} + \cancel{12i} - 9i^2$

$= x^2 - 4x - 4x + 16 - 9i^2$

$= x^2 - 8x + 16 - 9(-1)$

$= x^2 - 8x + 25$

$\boxed{x^2 - 8x + 25}$

c. repeated root can be expressed as $(x-\gamma)^2$ where γ is the repeated root.

Question 4 continued

$(x^2-8x+25)(x-r)^2$

$(x^2-8x+25)(x^2-2rx+r^2)$

$x^4-2rx^3+r^2x^2-8x^3+16rx^2-8r^2x+25x^2-50rx+25r^2$

(1) $x^4+(-2r-8)x^3+(r^2+16r+25)x^2+(-8r^2-50r)x+(25r^2)$

compare coefficients to eq^n above:
(1) $1=1$
(2) $A=-2r-8$
(3) $B=r^2+16r+25$
(4) $C=-8r^2-50r$
(5) $225=25r^2$

Use (5) to solve for r:
$225=25r^2$
$r^2=\frac{225}{25}$
$r^2=9$
$r=\pm\sqrt{9}=\pm3$

However, Q states r is positive $\therefore r=3$ only.

$\boxed{\text{real root} = 3}$

a. sub $r=3$ back into eq^{n}s (2) - (4):
(2) $A=-2(3)-8=-14$
(3) $B=(3)^2+16(3)+25=82$
(4) $C=-8(3)^2-50(3)=-222$

$\boxed{A=-14, B=82, C=-222}$

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5.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

The matrix **P** represents the transformation U

(a) Give a full description of U as a single geometrical transformation.
(2)

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line $y = -x$

(b) Write down the matrix **Q**
(1)

The transformation U followed by the transformation V is represented by the matrix **R**

(c) Determine the matrix **R**
(2)

The transformation W is represented by the matrix 3**R**

The transformation W maps a triangle T to a triangle T'

The transformation W' maps the triangle T' back to the original triangle T

(d) Determine the matrix that represents W'
(3)

a. $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ ← Anticlockwise rotation through ϑ about O (Given in Formulae booklet)

$$\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Equate the entries in the matrix : $\cos\vartheta = \frac{1}{2}$ $\quad \sin\vartheta = \frac{\sqrt{3}}{2}$

$$\frac{\sin\vartheta}{\cos\vartheta} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\tan\vartheta = \sqrt{3}$$

$$\vartheta = \tan^{-1}(\sqrt{3})$$

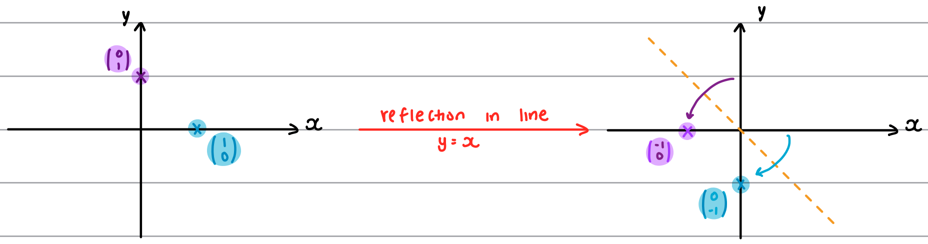
$$\vartheta = 60^\circ$$

Question 5 continued

$\boxed{\therefore \text{ Matrix } P \text{ represents an anticlockwise rotation through } 60^\circ \text{ (3 s.f.) about } O.}$

b. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then transform both of these coordinates by transformation (reflection in line $y = -x$)



identity matrix I

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \xrightarrow{\text{new matrix represents transformation}} \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$$

rewrite matrix but now replace coordinates with their transformations in this specific order

$$\boxed{Q : \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}}$$

c. order : $R = VU$

$$R = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

$$R = \begin{bmatrix}(0)(\frac{1}{2}) + (-1)(\frac{\sqrt{3}}{2}) & (0)(-\frac{\sqrt{3}}{2}) + (-1)(\frac{1}{2})\\(-1)(\frac{1}{2}) + (0)(\frac{\sqrt{3}}{2}) & (-1)(-\frac{\sqrt{3}}{2}) + (0)(\frac{1}{2})\end{bmatrix}$$

$$\boxed{R = \begin{bmatrix}-\frac{\sqrt{3}}{2} & -\frac{1}{2}\\-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}}$$

a. We are trying to find $(3R)^{-1}$

$$3R = \begin{bmatrix}-\frac{3\sqrt{3}}{2} & -\frac{3}{2}\\-\frac{3}{2} & \frac{3\sqrt{3}}{2}\end{bmatrix}$$

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Question 5 continued

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

$\det(3R) = \left(-\frac{3\sqrt{3}}{2}\right)\left(\frac{3\sqrt{3}}{2}\right) - \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)$

$= -9$

$\det(3R) = -9$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^{-1} = \frac{1}{\det(X)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -\left(-\frac{3}{2}\right) = 3/2$

$-c = -\left(-\frac{3}{2}\right) = 3/2$

$$(3R)^{-1} = \frac{-1}{9}\begin{bmatrix} \frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{bmatrix}$$

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6. The quadratic equation

$$Ax^2 + 5x - 12 = 0$$

where A is a constant, has roots α and β

(a) Write down an expression in terms of A for

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(2)

The equation

$$4x^2 - 5x + B = 0$$

where B is a constant, has roots $\alpha - \dfrac{3}{\beta}$ and $\beta - \dfrac{3}{\alpha}$

(b) Determine the value of A

(3)

(c) Determine the value of B

(3)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha+\beta)x + (\alpha\beta) = 0$

sum of roots: $\alpha + \beta$

product of roots: $\alpha\beta$

$Ax^2 + 5x - 12 = 0$ $\div A$

$x^2 + \frac{5}{A}x - \frac{12}{A} = 0$

$\alpha + \beta = -\frac{5}{A}$

$\alpha\beta = -\frac{12}{A}$

i. $\boxed{\alpha + \beta = -\frac{5}{A}}$

ii. $\boxed{\alpha\beta = -\frac{12}{A}}$

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Question 6 continued

b. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - \left(\alpha - \frac{3}{\beta} + \beta - \frac{3}{\alpha}\right)x + \left(\left(\alpha - \frac{3}{\beta}\right) \times \left(\beta - \frac{3}{\alpha}\right)\right) = 0$

$x^2 - \left(\alpha - \frac{3}{\beta} + \beta - \frac{3}{\alpha}\right)x + \left(\alpha\beta - 3 - 3 + \frac{9}{\alpha\beta}\right) = 0$

$x^2 - \left(\alpha - \frac{3}{\beta} + \beta - \frac{3}{\alpha}\right)x + \left(\alpha\beta + \frac{9}{\alpha\beta} - 6\right) = 0$

$4x^2 - 5x + B = 0$

$x^2 - \frac{5}{4}x + \frac{B}{4} = 0$

$\alpha - \frac{3}{\beta} + \beta - \frac{3}{\alpha} = \frac{5}{4}$

$\alpha + \beta - \frac{3}{\alpha} - \frac{3}{\beta} = \frac{5}{4}$

$(\alpha + \beta) - 3\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{5}{4}$

$(\alpha + \beta) - 3\left(\frac{\alpha + \beta}{\alpha\beta}\right) = \frac{5}{4}$

$\left(-\frac{5}{A}\right) - 3\left(\frac{\cancel{-}\frac{5}{\cancel{A}}}{\cancel{-}\frac{12}{\cancel{A}}}\right) = \frac{5}{4}$

$-\frac{5}{A} - 3\left(\frac{5}{12}\right) = \frac{5}{4}$

$-\frac{5}{A} = \frac{5}{4} + 3\left(\frac{5}{12}\right)$

$-\frac{5}{A} = \frac{5}{2}$

$-\frac{A}{5} = \frac{2}{5}$

$A = -2$

$\boxed{A = -2}$

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Question 6 continued

c. $\left(\alpha - \frac{3}{\beta}\right)\left(\beta - \frac{3}{\alpha}\right) = \frac{B}{4}$

$\alpha\beta - 3 - 3 + \frac{9}{\alpha\beta} = \frac{B}{4}$

$\frac{-12}{A} - 6 + \frac{9}{(-12/A)} = \frac{B}{4}$

$\frac{-12}{(-2)} - 6 + \frac{9}{(-12/-2)} = \frac{B}{4}$

$6 - 6 + \left(\frac{3}{2}\right) = \frac{B}{4}$

$\frac{3}{2} = \frac{B}{4}$

$3(\cancel{4}^{2}) = B(\cancel{2}^{1})$

$6 = B$

$\boxed{B = 6}$

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7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The rectangular hyperbola H has equation $xy = 36$

The point $P(4, 9)$ lies on H

(a) Show, using calculus, that the normal to H at P has equation

$$4x - 9y + 65 = 0$$

(4)

The normal to H at P crosses H again at the point Q

(b) Determine an equation for the tangent to H at Q, giving your answer in the form $y = mx + c$ where m and c are rational constants.

(5)

a. Differentiate rectangular hyperbola eq^n w.r.t. x:

$xy = 36$

$y = \frac{36}{x} = 36x^{-1}$

$\frac{dy}{dx} = -36x^{-2} = \frac{-36}{x^2}$

Sub in point P (4,9):

$\left.\frac{dy}{dx}\right|_{(4,9)} \quad \frac{-36}{(4)^2} = \frac{-9}{4}$

$M_{tangent} = -\frac{9}{4}$

$M_{normal} = \frac{4}{9}$

set up line eq^n $y - y_1 = m(x - x_1)$ with $m = \frac{4}{9}$

$(x_1, y_1) = (4, 9)$

$y - 9 = \frac{4}{9}(x - 4)$

$9(y - 9) = 4(x - 4)$

$9y - 81 = 4x - 16$

$\boxed{4x - 9y + 85 = 0 \text{ // shown}}$

b. Firstly, find coord. Q - this is by solving $4x - 9y + 65 = 0$ and $xy = 36$ simultaneously.

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Question 7 continued

(1) $4x - 9y + 65 = 0$

(2) $xy = 36$

(1) $4x - 9y + 65 = 0$ rearrange for $4x$

$4x = 9y - 65$

(2) $xy = 36$ x4 on both sides

$4(xy) = 4(36) \Rightarrow 4xy = 144$

$4xy = 144$ sub $4x = 9y - 65$ into $4xy = 144$

$(9y-65)y = 144$

$9y^2 - 65y = 144$

$9y^2 - 65y - 144 = 0$

$(9y+16)(y-9) = 0$

$y = -\frac{16}{9}$ or $y = 9$

y-coord. of Q. belongs to point P.

now sub $y = -\frac{16}{9}$ into either (1) or (2) to find x-coord. of Q.

$xy = 36$

$x\left(-\frac{16}{9}\right) = 36$

$x = \frac{36}{-16/9} = \frac{-81}{4}$

$\therefore Q: \left(\frac{-81}{4}, \frac{-16}{9}\right)$

Now use coord. Q to find tangent to H @ point Q

Differentiate rectangular hyperbola eq^n w.r.t. x:

$xy = 36$

$y = \frac{36}{x} = 36x^{-1}$

$\frac{dy}{dx} = -36x^{-2} = \frac{-36}{x^2}$

sub in point $Q\left(\frac{-81}{4}, \frac{-16}{9}\right)$:

$\left.\frac{dy}{dx}\right|_{\left(-\frac{81}{4}, -\frac{16}{9}\right)} \quad \frac{-36}{\left(\frac{-81}{4}\right)^2} = \frac{-64}{729}$

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Question 7 continued

$M_{tangent} : -\frac{64}{729}$

set up line eq^n $y - y_1 = m(x - x_1)$ with $m = \frac{-64}{729}$

$(x_1, y_1) = \left(\frac{-81}{4}, \frac{-16}{9}\right)$

$y - \left(-\frac{16}{9}\right) = \frac{-64}{729}\left(x - \left(\frac{-81}{4}\right)\right)$

$y + \frac{16}{9} = -\frac{64}{729}\left(x + \frac{81}{4}\right)$

$y + \frac{16}{9} = \frac{-64}{729}x - \frac{16}{9}$

$y = \frac{-64}{729}x - \frac{32}{9}$

$\boxed{y = -\frac{64}{729}x - \frac{32}{9} \,/\!/ \quad m = \frac{-64}{729}, \; c = \frac{-32}{9}}$

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8. $$f(x) = 2x^{-\frac{2}{3}} + \frac{1}{2}x - \frac{1}{3x-5} - \frac{5}{2} \qquad x \neq \frac{5}{3}$$

The table below shows values of f(x) for some values of x, with values of f(x) given to 4 decimal places where appropriate.

x	1	2	3	4	5
f(x)	0.5	−1.2401	−0.2885	0.1508	0.5834

(a) Complete the table giving the values to 4 decimal places.
(2)

The equation f(x) = 0 has exactly one positive root, α.

Using the values in the completed table and explaining your reasoning,

(b) determine an interval of width one that contains α.
└ looking for 1 interval, not 2.
(2)

(c) Hence use interval bisection twice to obtain an interval of width 0.25 that contains α.
(3)

Given also that the equation f(x) = 0 has a negative root, β, in the interval [–1, –0.5]

(d) use linear interpolation once on this interval to find an approximation for β.

Give your answer to 3 significant figures.
(3)

a. $f(2) = 2(2)^{-\frac{2}{3}} + \frac{1}{2}(2) - \frac{1}{3(2)-5} - \frac{5}{2} = -1.24007895$

$f(4) = 2(4)^{-\frac{2}{3}} + \frac{1}{2}(4) - \frac{1}{3(4)-5} - \frac{5}{2} = 0.1508483831$

$\boxed{f(2) = -1.2401 \text{ (4 d.p.)} \quad f(4) = 0.1508 \text{ (4 d.p.)}}$

b. there is a sign change in interval [1,2] from positive to negative

there is also a sign change in interval [3,4] from negative to positive.

In order for there to be a root in the interval, there must be a sign change but also the function must be continuous in that interval.

Question 8 continued

$f(x) = 2x^{-\frac{2}{3}} + \frac{1}{2}x - \frac{1}{3x-5} - \frac{5}{2}$

$\frac{1}{3x-5}$, $3x - 5 \neq 0$

$3x - 5 \neq 0$

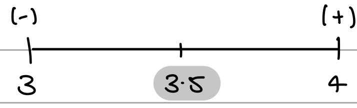
$3x \neq 5$

$x \neq \frac{5}{3}$

$f(x)$ is undefined at $x = \frac{5}{3}$ $\therefore$ [1,2] is not a suitable interval.

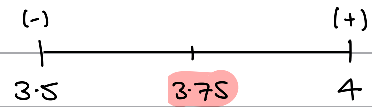
$\boxed{\therefore \text{ root}, \alpha, \text{ lies in } [3,4]}$

c. 1^{st} interval bisection : $\frac{3 + 4}{2} = 3.5$

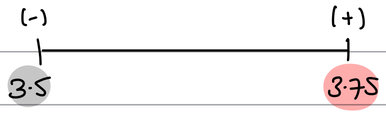


$f(3.5) = -0.06422133271$

2^{nd} interval bisection : $\frac{3.5 + 4}{2} = 3.75$



$f(3.75) = 0.04359533492$



$\boxed{3.5 \leq \alpha \leq 3.75}$

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Question 8 continued

d. Linear Interpolation:

$$\alpha = \frac{af(b) - bf(a)}{f(b) - f(a)}$$ for interval $[a,b]$

where α is the root

$a = -1 \qquad b = -0.5$

$$\beta = \frac{-1\,f(-0.5) - -0.5\,f(-1)}{f(-0.5) - f(-1)}$$

$f(-1) = 2(-1)^{-\frac{2}{3}} + \frac{1}{2}(-1) - \frac{1}{3(-1)-5} - \frac{5}{2} = -\frac{7}{8}$

$f(-0.5) = 2(-0.5)^{-\frac{2}{3}} + \frac{1}{2}(-0.5) - \frac{1}{3(-0.5)-5} - \frac{5}{2} = 0.578648257 8$

$$\beta = \frac{-1(0.5786482578) + 0.5\left(-\frac{7}{8}\right)}{(0.5786482578) - \left(-\frac{7}{8}\right)}$$

$\beta = -0.6990331068$

$\boxed{\beta = -0.699 \ (3 \text{ s.f.})}$

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9. (a) Prove by induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the standard summation formulae, show that

$$\sum_{r=1}^{n} r(r+1)(r-1) = \frac{1}{4}n(n+A)(n+B)(n+C)$$

where A, B and C are constants to be determined.

(4)

(c) Determine the value of n for which

$$3\sum_{r=1}^{n} r(r+1)(r-1) = 17\sum_{r=n}^{2n} r^2$$

(5)

a. (1) Base case: Prove true for $n = 1$

LHS: $\sum_{r=1}^{1} r^3 = (1)^3 = 1$

RHS: $\frac{1}{4}(1)^2(1+1)^2 = 1$

LHS = RHS

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$$

(3) Prove true for $n = k+1$:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

we are rewriting sum of $(k+1)$ terms as the sum of (k) terms + $(k+1)^{th}$ term – we can now use step (2).

Thinking ahead, sub in $n = k+1$ into RHS: this is what we are trying to prove using (2):

$$\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+1+1)^2$$

Make a note so we know what we are working towards

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right]$$

$$= \frac{1}{4}(k+1)^2\left[k^2 + 4k + 4\right]$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \frac{1}{4}(k+1)(k+1+1)^2$$

$\therefore$ true for $n = k+1$

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Question 9 continued

(4) Conclusion

Hence result is true for $n=k+1$. As true for $n=1$ and have shown if true for $n=k$ then it is true for $n=k+1$, so it is true for all $n \in \mathbb{N}$ by induction.

b. $\sum_{r=1}^{n} r(r+1)(r-1) = \sum_{r=1}^{n} r^3 - r$

split summation into 2

$\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

in the formulae booklet

$\frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)$

factor out $\frac{1}{4}n(n+1)$

$\frac{1}{4}n(n+1)[n(n+1) - 2]$

$\frac{1}{4}n(n+1)[n^2+n-2]$

$\boxed{\frac{1}{4}n(n+1)(n+2)(n-1) \quad A=1 \quad B=2 \quad C=-1}$

c. $3\sum_{r=1}^{n} r(r+1)(r-1) = 17\sum_{r=n}^{2n} r^2$

LHS: $3\sum_{r=1}^{n} r(r+1)(r-1) = 3 \times \frac{1}{4}n(n+1)(n+2)(n-1)$

$= \frac{3}{4}n(n+1)(n+2)(n-1)$

RHS: $17\sum_{r=n}^{2n} r^2 = 17\left[\sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2\right]$

$= 17\left[\frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}(n-1)(n-1+1)(2(n-1)+1)\right]$

$= 17\left[\frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}(n-1)(n)(2n-1)\right]$

factor out $\frac{1}{6}n$

$= \frac{17}{6}n\left[2(2n+1)(4n+1) - (n-1)(2n-1)\right]$

$= \frac{17}{6}n\left[2(8n^2+6n+1) - (2n^2-3n+1)\right]$

$= \frac{17}{6}n\left[16n^2+12n+2-2n^2+3n-1\right]$

$= \frac{17}{6}n\left[14n^2+15n+1\right]$

$= \frac{17}{6}n(14n+1)(n+1)$

P 7 1 0 9 9 A 0 3 3 3 6

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Question 9 continued

equate LHS = RHS now.

$\frac{3}{4}n(n+1)(n+2)(n-1) = \frac{17}{6}n(14n+1)(n+1)$

$\frac{3}{4}n(n+1)(n+2)(n-1) - \frac{17}{6}n(14n+1)(n+1) = 0$

factor out $\frac{1}{12}n(n+1)$

$\frac{1}{12}n(n+1)\left[9(n+2)(n-1) - 34(14n+1)\right] = 0$

$\frac{1}{12}n(n+1)\left[9(n^2+n-2) - 476n - 34\right] = 0$

$\frac{1}{12}n(n+1)\left[9n^2+9n-18 - 476n - 34\right] = 0$

$\frac{1}{12}n(n+1)\left[9n^2 - 467n - 52\right] = 0$

$\frac{1}{12}n(n+1)(9n+1)(n-52) = 0$

$\therefore n = 0,\ n = -1,\ n = -\frac{1}{9},$ or, $n = 52$

n must be a positive integer, $\therefore n = 52$

$\boxed{n = 52}$

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